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The Cracked Shear Panel

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A centrally cracked sheet subject to shear loads, i.e., a cracked shear panel, is analyzed in detail. The standard one-term approximation for the elastic stress and displacement local to the crack tip region is shown in several respects to be inadequate. Two different approaches are used to determine the global elastic strain energy rate. One of the methods, apart from serving to corroborate the results of the other, also shows how the elastic strain energy rate can be decomposed into two contributions—one corresponding to the rate of mechanical work of the applied boundary tractions, the other corresponding to the rate of mechnical work performed by the stresses in the process of crack extension at both crack ends. The fracture stress is also determined in accordance with Griffith's crack instability hypothesis.

Nomenclature

$2a$	= crack dimension
C_1, C_2	= simple closed boundary curves of R
E	= Young's elastic modulus
K_I, K_{II}	= elastic stress intensity factors for the opening and sliding modes, respectively
r, θ	= polar coordinates
R	= bound region of the x-y plane
t_{xx}, t_{yy}, t_{xy}; $t'_{xx}, t'_{yy}, t'_{xy}$	= rectangular stress components
$t_{\theta\theta}$	= tangential stress component
$u_x, u_y; u'_x, u'_y$	= rectangular displacement components
ΔU	= change of elastic strain energy associated with formation of the crack $2a$
U_0	= elastic strain energy of the sheet without the crack
U, U'	= elastic strain energy per unit thickness
$x, y; x^*, y^*$	= plane rectangular coordinates
z	$= x + iy$, complex variable
ζ	$= z - a = re^{i\theta}$, complex variable
β	= crack orientation angle
θ_0	= angle of initial crack extension
τ	= applied boundary shear traction
τ_m	= maximum shear stress
τ^*_m	= maximum shear stress
τ_{cr}	= critical or fracture applied shear traction
$\Phi, \Omega, \phi, \omega$	= sectionally holomorphic functions of the complex variables z, ζ
Γ	= surface energy
γ	= surface energy density per unit area
μ	= elastic shear modulus
ν	= Poisson's ratio
κ	$= 3\text{-}4\nu$ for plane strain, $(3\text{-}\nu)/(1+\nu)$ for plane stress

I. Introduction

THE centrally cracked sheet or plate, the distant outer boundaries of which are loaded by pure shear, serves as a prototype or model having interest in aerospace applications (cf. Fig. 1). Although this problem has been the subject of previous analysis,[1] that part pertaining to the region very close to the crack tip can be shown to have been treated inadequately. At the global level, the elastic strain energy rate, and from it the critical value of the applied shear stress, have also been calculated; unfortunately, with differing results.[2-4] Recent theoretical study of the observed effects of loads applied parallel to a crack (i.e., of biaxial loading) on elastic fracture behavior,[5-8] effects which according to accepted theory are not supposed to exist,[1,2,9,10] has brought to light certain analytical inadequacies often employed in fracture mechanics—shortcomings that are avoided in the analysis of the shear panel given here.

II. Exact Solution

The plane components of the stress tensor relative to the pair of rectangular coordinates shown in Fig. 1 are related by

$$t_{yy} - t_{xx} + 2it_{xy} = e^{2i\beta}(t^*_{yy} - t^*_{xx} + 2it^*_{xy}) \tag{1}$$

Since the first principal invariant of the stress tensor has the same value in all coordinate systems, we have, for both plane stress and plane strain,

$$t_{xx} + t_{yy} = t^*_{xx} + t^*_{yy} \tag{2}$$

Along the remote outer boundaries of the body relative to the $(x^* - y^*)$ coordinate system

$$t^*_{xx}(\infty) = t^*_{yy}(\infty) = 0, \quad t^*_{xy}(\infty) = \tau \tag{3}$$

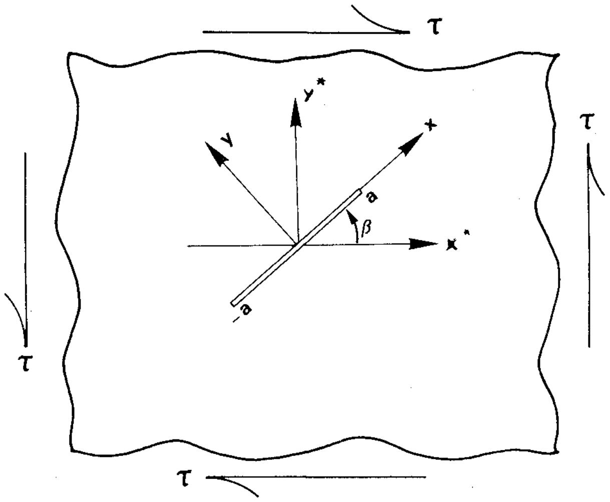

Fig. 1 Cracked shear panel.

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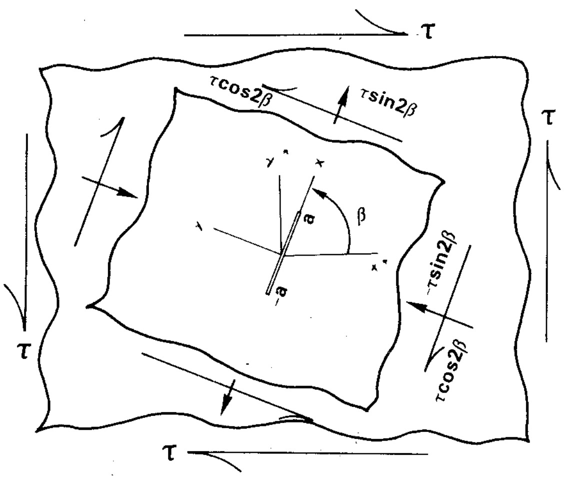

Fig. 2 Boundary conditions relative to the x-y coordinate system.

From Eqs. (1-3) the boundary conditions relative to the (x-y) coordinate system can be directly calculated to be (cf. Fig. 2):

$$t_{xx}(\infty)=\tau\sin 2\beta,\ t_{yy}(\infty)=-\tau\sin 2\beta,\ t_{xy}(\infty)=\tau\cos 2\beta \quad (4)$$

Neglecting rigid body rotation at remote points of the body, the solution to the problem posed by Fig. 2, i.e., for boundary conditions of Eq. (4), is given by the following pair of sectionally holomorphic functions of the complex variable z (Ref. 11):

$$\left.\begin{array}{l}\Phi(z)\\ \Omega(z)\end{array}\right\}=-\frac{1}{2}i\tau e^{-2i\beta}\left[\frac{z}{\sqrt{z^2-a^2}}\mp 1\right] \quad (5)$$

assuming that the boundaries of the cut (line crack) are free of traction‡

$$t_{yy}(x,o)=t_{xy}(x,o)=0 \quad |x|<a \quad (6)$$

The positive branch of the square root can be taken on the positive x axis: $x>a$. The components of stress and displacements are specified in terms of Φ and Ω through the relations

$$t_{yy}-t_{xx}+2it_{xy}=2[(\bar{z}-z)\Phi'(z)+\bar{\Omega}(z)-\Phi(z)]$$
$$t_{yy}+t_{xx}=2[\Phi(z)+\overline{\Phi(z)}] \quad (7)$$
$$2\mu(u_x+iu_y)=\kappa\phi(z)-\omega(\bar{z})-(z-\bar{z})\overline{\Phi(z)}$$

where

$$\phi(z)=\int\Phi(z)\mathrm{d}z \quad \omega(z)\int\Omega(z)\mathrm{d}z \quad (8)$$

The overbar represents complex conjugation and the prime differentiation. From Eqs. (5) and (8),

$$\left.\begin{array}{l}\phi(z)\\ \omega(z)\end{array}\right\}=-\frac{1}{2}i\tau e^{-2i\beta}[\sqrt{z^2-a^2}\mp z] \quad (9)$$

to within an arbitrary complex constant that represents negligible rigid body displacements.

The complex elastic stress intensity factor corresponding to opening (closing) and sliding modes of crack border displacements is determined entirely by the function Φ, i.e.,

$$K_I-iK_{II}=\lim_{z\to a}[2\sqrt{2\pi}\sqrt{(z-a)}\Phi(z)]=-i\tau\sqrt{\pi a}\,e^{-2i\beta} \quad (10)$$

from which

$$K_I=-\tau\sqrt{\pi a}\sin 2\beta,\quad K_{II}=\tau\sqrt{\pi a}\cos 2\beta \quad (11)$$

III. Stress and Displacement Approximations Near the Crack Tip

The transformation $z=\zeta+a=re^{i\theta}+a$ introduces a polar coordinate system at the crack tip (cf., Fig. 3), whereby

$$\left.\begin{array}{l}\Phi(\zeta)\\ \Omega(\zeta)\end{array}\right\}=-\frac{1}{2}i\tau e^{-2i\beta}\left[\frac{(\zeta+a)}{\sqrt{\zeta^2+2a\zeta}}\mp 1\right] \quad (12)$$

Expanding Φ, and thereby Ω, as power series about the origin

$$\Phi(\zeta)=-\frac{1}{2}i\tau e^{-2i\beta}\frac{1}{\sqrt{2}}\left[\left(\frac{\zeta}{a}\right)^{-1/2}-\sqrt{2}\left(\frac{\zeta}{a}\right)^{0}+\frac{3}{4}\left(\frac{\zeta}{a}\right)^{1/2}-\frac{5}{32}\left(\frac{\zeta}{a}\right)^{3/2}+\ldots\right] \quad (13)$$

which, with the help of Eq. (10), may be expressed in the form

$$\Phi(\zeta)=\frac{(K_I-iK_{II})}{2\sqrt{2\pi\zeta}}+\frac{1}{2}i\tau e^{-2i\beta}+O\left(\left|\frac{\zeta}{a}\right|^{1/2}\right) \quad (14)$$

The last term indicates that the remainder of the series is of the order $(r/a)^{1/2}$ and higher. When $|\zeta|=r$ is restricted to very small positive values relative to half the crack size, the higher-order terms may be ignored with the resulting approximations

$$\left.\begin{array}{l}2\Phi(\zeta)\\ 2\Omega(\zeta)\end{array}\right\}=\frac{(K_I-iK_{II})}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)\pm i\tau e^{-2i\beta} \quad (15)$$

Because of the singular nature of the first term of the expansion, Eq. (13), it is customary in fracture mechanics to retain only this term in determining a suitable approximation for the stress components on the grounds that this contribution dominates, quantatively speaking, as long as the ratio (r/a) is kept sufficiently small. Note, however, that the second term of the expansion, Eq. (13), and its approximation, Eq. (15), is independent of distance from the crack tip origin. Failure to include this contribution can lead to significant error in describing the mechanial situation in the vicinity surrounding the crack tip, as will presently be shown. As a matter of fact, for a sheet with a horizontal crack loaded biaxially in tension-tension or tension-compression, it is precisely this second nonsingular term that accounts entirely for the effect of the load applied parallel to the crack. It was

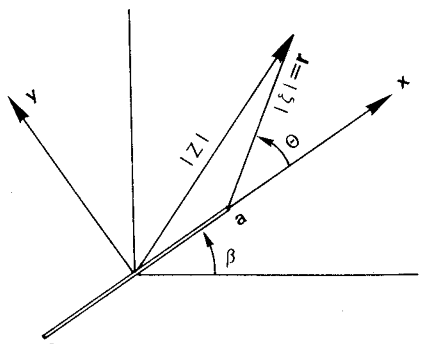

Fig. 3 Polar coordinate system at the crack tip.

‡This condition can be satisfied or, equivalently, open-mode crack surface displacements will be assured if the angle β is restricted to the fourth quadrant, i.e., $0\leq-\beta\leq\pi/2$, as may be deduced from Eq. (21).

through neglect of this term that much of the misinformation claiming indifference of fracture behavior to load biaxiality came to pass. Upon substitution of Eq. (15) into Eq. (7), it follows that

$$t_{yy} \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \tag{16a}$$

$$t_{xx} \approx \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] + 2\tau\sin\beta \tag{16b}$$

$$t_{xy} \approx \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \tag{16c}$$

provided $0 < (r/z) \ll 1$. Retention of the second nonsingular term of the expansion, Eq. (13), results in the presence of the $2\tau\sin 2\beta$ term in the approximation for t_{xx}.

In a similar manner, series expansion of

$$\left.\begin{matrix}\phi(\zeta)\\ \omega(\zeta)\end{matrix}\right\} = -\frac{1}{2} i\tau e^{-2i\beta}\left[\sqrt{\zeta^2 + 2a\zeta} \mp (\zeta + a)\right] \tag{17}$$

gives

$$\phi(\zeta) = -\frac{1}{2} i\tau e^{-2i\beta} a\sqrt{2}\left\{\left[\left(\frac{\zeta}{a}\right)^{1/2} - \frac{1}{\sqrt{2}}\left(\frac{\zeta}{a}\right) + \frac{1}{2}\left(\frac{\zeta}{a}\right)^{3/2} - \dots\right] - \frac{1}{\sqrt{2}}\right\} = (K_I - iK_{II})\sqrt{\frac{\zeta}{2\pi}} + \frac{1}{2} i\tau a\left(\frac{\zeta}{a} + 1\right)e^{-2i\beta} + O\left(\left|\frac{\zeta}{a}\right|^{3/2}\right) \tag{18}$$

If the approximation for the displacements is to be consistent with that taken for the stress, terms of order $(r/a)^{3/2}$ and higher in the expansion can be discarded, leaving

$$\left.\begin{matrix}\phi(\zeta)\\ \omega(\zeta)\end{matrix}\right\} \approx (K_I - iK_{II})\sqrt{\frac{r}{2\pi}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \pm \frac{1}{2} i\tau a\left(\frac{r}{a}e^{i\theta} + 1\right)e^{-2\beta} \tag{19}$$

Approximate expressions for the displacements local to the crack tip follow from Eqs. (19) and (7) as

$$u_x \approx \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[\frac{(\kappa-1)}{2} + \sin^2\frac{\theta}{2}\right] + \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\frac{(\kappa+1)}{2} + \cos^2\frac{\theta}{2}\right] + \frac{\tau}{4\mu}\left[(\kappa+1)(r\cos\theta + a)\sin 2\beta - (\kappa+3)r\sin\theta\cos 2\beta\right] \tag{20a}$$

$$u_y \approx \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\frac{(\kappa+1)}{2} - \cos^2\frac{\theta}{2}\right] + \frac{K_{II}}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[\frac{(1-\kappa)}{2} + \sin^2\frac{\theta}{2}\right] + \frac{\tau}{4\mu}\left[(\kappa+1)(r\cos\theta + a)\cos 2\beta + (\kappa-5)r\sin\theta\sin 2\beta\right] \tag{20b}$$

The nonsingular second term of expansion, Eq. (18) manifests itself through the second lines of Eqs. (20), which, of course, do not appear in the standard expressions for displacements in the crack-tip region. At the tip of the crack

$$[u_x]_{r=0} = \frac{\tau(\kappa+1)a}{4\mu}\sin 2\beta \tag{21a}$$

$$[u_y]_{r=0} = \frac{\tau(\kappa+1)a}{4\mu}\cos 2\beta \tag{21b}$$

According to the conventional approximation, both displacement components would be predicted to be zero at the crack tip.

IV. Maximum Shear Stress

For plane states of stress, the square of the maximum shear stress is determined by

$$\tau_m^2 = \tfrac{1}{4}(t_{yy} - t_{xx})^2 + t_{xy}^2 \tag{22}$$

Thus, in the immediate vicinity of the crack tip, Eqs. (16) and (22) combine to yield

$$\tau_m^2 \approx \frac{1}{8\pi r}\left[K_I^2\sin^2\theta + K_{II}^2(4 - 3\sin^2\theta) + 4K_I K_{II}\sin\theta\cos\frac{\theta}{2}\right] - \frac{\tau\sin 2\beta}{\sqrt{2\pi r}}\left[K_I\sin\theta\sin\frac{3\theta}{2} + K_{II}\left(\sin\theta\cos\frac{3\theta}{2} + 2\sin\frac{\theta}{2}\right)\right] - \tau^2\sin^2 2\beta \tag{23}$$

If only the customary singular approximation for the stress were to be employed in Eq. (22), the result would be that given

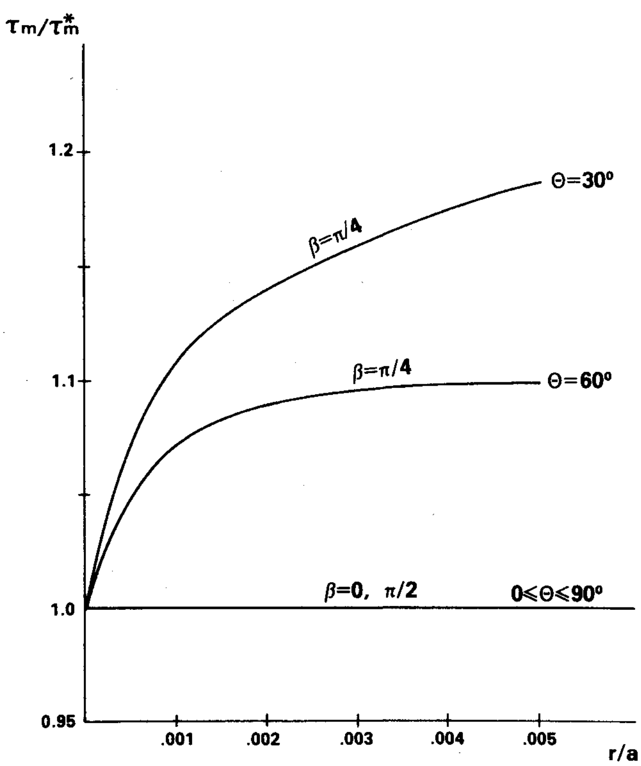

Fig. 4 Error in conventional maximum shear stress calculation near the crack tip.

by the first line of Eq. (23), i.e.,

$$\tau_m^{*2} \approx \frac{1}{8\pi r}\left[K_I^2\sin^2\theta + K_{II}^2(4-3\sin^2\theta) + 4K_IK_{II}\sin\theta\cos\frac{\theta}{2}\right] \quad (24)$$

A measure of the error involved in the use of Eq. (24) as against Eq. (23) can be evaluated by the ratio

$$\left(\frac{\tau_m}{\tau_m^*}\right)^2 = 1 - \sqrt{32\left(\frac{r}{a}\right)}$$

$$\times \frac{\left[\cos2\beta\left(\sin\theta\cos\frac{3\theta}{2} + 2\sin\frac{\theta}{2}\right) - \sin2\beta\sin\theta\sin\frac{3\theta}{2}\right]}{\{\sin2\beta\sin^2\theta + \cot2\beta\cos2\beta(4-3\sin^2\theta) - 4\cos2\beta\sin\theta\cos(\theta/2)\}}$$

$$-8\left(\frac{r}{a}\right)\frac{1}{[\sin^2\theta + \cot^2 2\beta(4-3\sin^2\theta) - 4\cot2\beta\sin\theta\cos(\theta/2)]} \quad (25)$$

Figure 4 provides some idea of the extent of the difference which, as seen, is not insignificant.

Equation (23) can be rearranged to the following form, which is more convenient for use in plotting isostats of normalized maximum shear stress, examples of which are shown in Figs. 5-8.

$$\left[\left(\frac{\tau_m}{\tau}\right)^2 + \sin2\beta\right]\left(\frac{r}{a}\right) - \frac{1}{\sqrt{2}}\left[\sin^2 2\beta F_1(\theta) - \sin2\beta\cos2\beta F_2(\theta)\right]\left(\frac{r}{a}\right)^{1/2} - \frac{1}{8}[\sin^2 2\beta F_3(\theta) + \cos^2 2\beta F_4(\theta) - 4\sin2\beta\cos2\beta F_5(\theta)] = 0 \quad (26)$$

where

$$F_1(\theta) = \sin\theta\sin\frac{3\theta}{2}, \quad F_2(\theta) = \sin\theta\cos\frac{3\theta}{2} + 2\sin\frac{\theta}{2}$$

$$F_3(\theta) = \sin^2\theta, \quad F_4(\theta) = 4 - 3\sin^2\theta, \quad F_5(\theta) = \sin\theta\cos\frac{\theta}{2}$$

V. Angle of Initial Crack Extension

The maximum tensile stress criterion will be employed to predict the angle at which the crack will just begin to extend. While this criterion suffers limitations due to oversimplification, it nevertheless has been shown to offer reasonably good correlation with experiment under tensile loadings.[12,13] Designating by θ_0 the angle which locates the tangent to the extended crack surface relative to the plane of the original crack, the maximum tensile stress hypothesis predicts that a crack will extend from its end along the radial line that is normal to the direction of the maximum tensile stress. Mathematically, this criterion applied to a line crack requires that at some small but unspecified distance r_0 from the tip of the crack, it is necessary and sufficient that θ_0 be such that the following three conditions be simultaneously satisfied:

$$[t_{\theta\theta}]_{\theta_0} > 0, \quad \left[\frac{\partial t_{\theta\theta}}{\partial\theta}\right]_{\theta_0} = 0, \quad \left[\frac{\partial^2 t_{\theta\theta}}{\partial\theta^2}\right]_{\theta_0} < 0 \quad (27)$$

The unspecified condition $r_0 > 0$ is necessitated by the singular nature of the tangential stress $t_{\theta\theta}$ at $r=0$, and thus introduces an element of approximation into the prediction of θ_0. It should also be observed that no elastic material parameters appear in the equations for the stress components in the plane

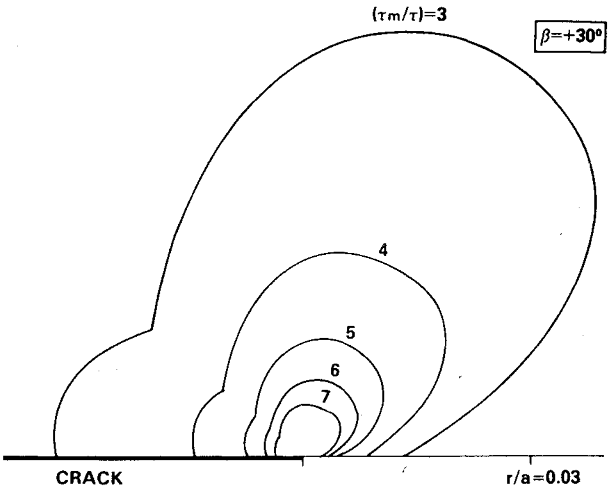

Fig. 5 Isostats of normalized maximum shear stress.

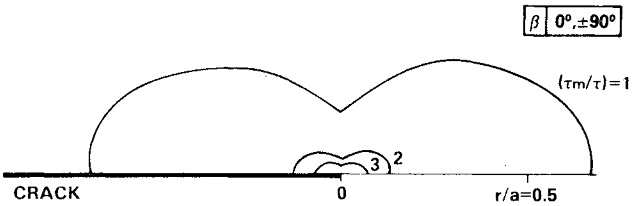

Fig. 6 Isostats of normalized maximum shear stress.

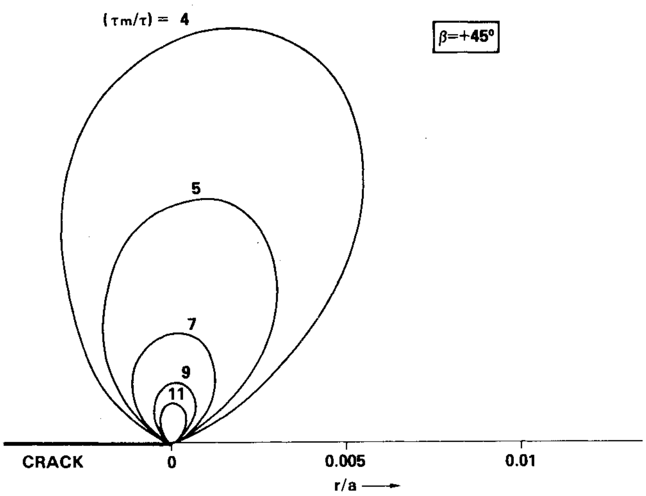

Fig. 7 Isostats of normalized maximum shear stress.

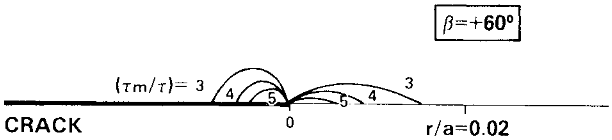

Fig. 8 Isostats of normalized maximum shear stress.

theory of linear elasticity in which body forces are ignored. Consequently, the above criterion when applied cannot distinguish one material from another in that it predicts the same values for θ_0 for all materials.

Using Eq. (11), it may be shown that[5] in the approximation that omits terms of order $(r/a)^{1/2}$ and above, consistent with the approximation that yields Eqs. (16),

$$\frac{t_{\theta\theta}}{\tau} \approx -\left[\frac{1}{32}\left(\frac{a}{r}\right)\right]^{1/2}\left[3\cos2\beta\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right)\right.$$

$$+\sin 2\beta\left(3\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\right)\Big]+2\sin 2\beta\sin^2\theta \qquad (28)$$

For some positive but arbitrarily small value $r_0>0$, conditions (27) will, in conjunction with Eq. (28), lead to the following set of relations specifying θ_0:

$$\alpha\left\{3\cos 2\beta\left[\sin\frac{\theta_0}{2}+\sin\frac{3\theta_0}{2}\right]+\sin 2\beta\left[3\cos\frac{\theta_0}{2}+\cos\frac{3\theta_0}{2}\right]\right\}-2\sin 2\beta\sin^2\theta_0<0 \qquad (29)$$

$$\alpha\left\{3\sin 2\beta\sin\theta_0-3\cos 2\beta[3\cos\theta_0-1]+8\sin 2\beta\sin\frac{\theta_0}{2}\cos\theta_0=0\right. \qquad (30)$$

$$\alpha\left\{3\cos 2\beta\sin\frac{\theta_0}{2}\left[9\sin^2\frac{\theta_0}{2}-7\right]+\sin 2\beta\cos\frac{\theta_0}{2}\left[6-9\cos^2\frac{\theta_0}{2}\right]\right\}-4\sin 2\beta\cos 2\theta_0>0 \qquad (31)$$

where $\alpha=[(1/32)(a/r)]^{1/2}$.

The values of θ_0 satisfying Eqs. (29-31) may be easily obtained graphically by location of the maximum of the $(t_{\theta\theta}/\tau)$ vs θ curves shown in Fig. 9 for the particular choice $(r_0/a)=0.01$. The variation of θ_0 with inclination of the crack is shown in Fig. 10, together with experimental values for 7075 and 2024 aluminum alloys obtained by Liu.[14]

VI. Global Elastic Strain Energy Rate

Calculation of the critical shear stress τ_{cr} necessary to induce fracture in accordance with Griffith's crack instability hypothesis appears to have been performed first by Swedlow[4] in 1965. Shortly thereafter, there followed a different analysis by Sih and Liebowitz,[2] resulting in a different expression for τ_{cr}. Since experimental information is lacking, the question of the validity of one or either of these results remains as yet unsettled. To determine the breaking stress, it is necessary first to know the rate at which the elastic strain energy of the entire body changes with crack size. The analysis given here utilizes alternative approaches, following along the lines of our previous work for biaxially applied loads.[8]

The elastic strain energy per unit thickness over an arbitrarily bound region R of the x-y plane is obtained by means of the double integral

$$U(R)=\iint_R\left[t_{xx}\frac{\partial u_x}{\partial x}+t_{yy}\frac{\partial u_y}{\partial y}+t_{xy}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\right]\mathrm{d}x\mathrm{d}y \qquad (32)$$

The region of the problem is the infinite plane separated along the cut (crack). It is convenient, therefore, to convert Eq. (32) to line integration by means of Green's theorem (for the multiply connected region) and the equations of equilibrium, with the result

$$U(R)=I_1+I_2=\frac{1}{2}\int_{C_1}[(t_{xx}u_x+t_{xy}u_y)\mathrm{d}y-(t_{yy}u_y+t_{xy}u_x)\mathrm{d}x]+\frac{1}{2}\int_{C_2}[(t_{xx}u_x+t_{xy}u_y)\mathrm{d}y-(t_{yy}u_y+t_{xy}u_x)\mathrm{d}x] \qquad (33)$$

where C_1 and C_2 are simple, but otherwise arbitrary, closed curves bounding region R (cf. Fig. 11). The strain energy for the entire plane is obtained in the limit as C_1 is made infinitely large, while C_2 is chosen to coincide with the contour of the crack. Proceeding with the later case first, the line integral along the boundary of the crack C_2 vanishes b virtue of the boundary conditon (6).

$$I_2=0 \qquad (34)$$

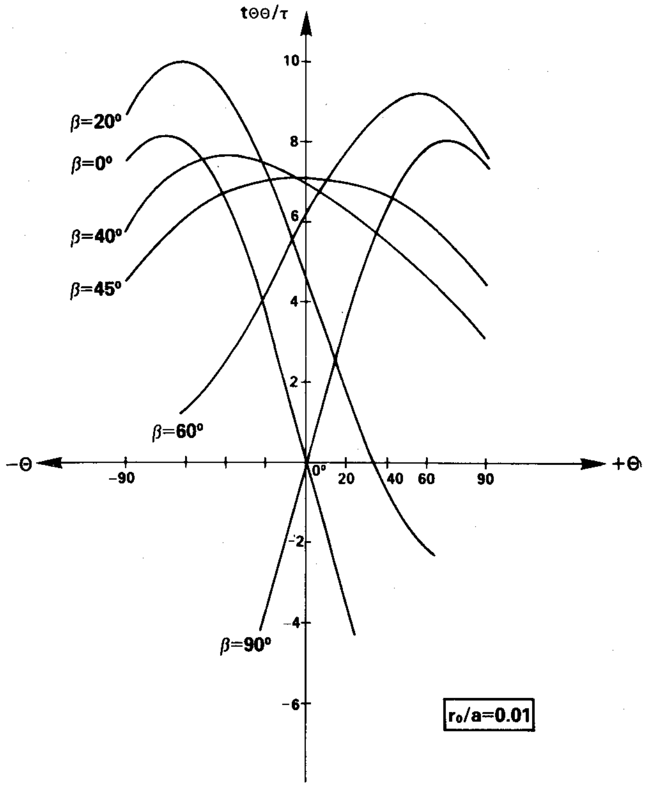

Fig. 9 Variation of normalized tangential stress with direction.

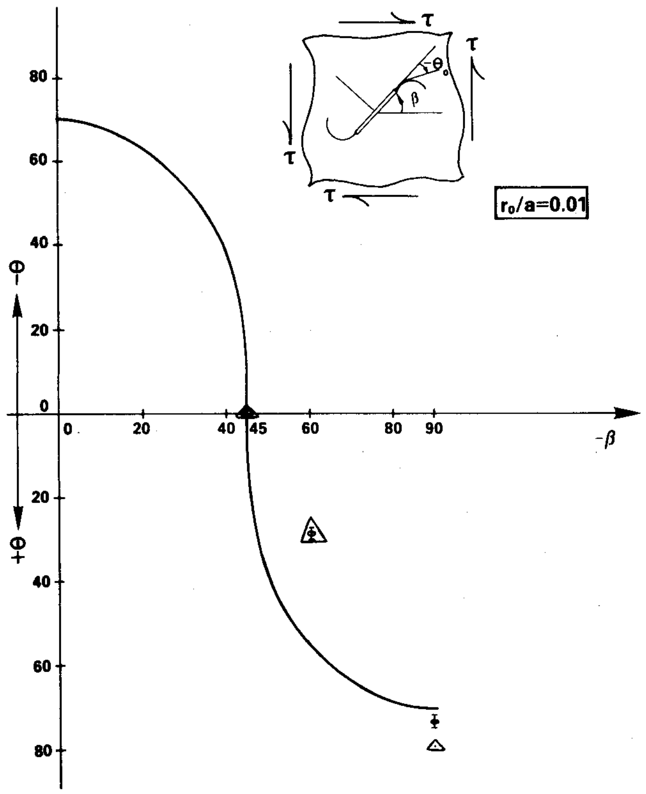

Fig. 10 Variation of angle of initial crack extension with angle of crack inclination.

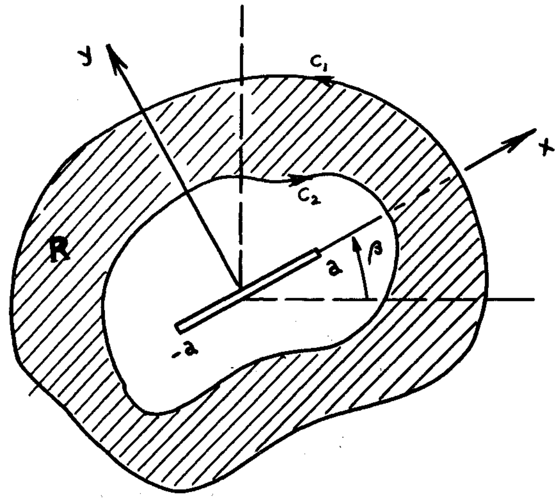

Fig. 11 Bound region R.

The outer boundary C_1 can be chosen to be a circle of radius $r_0 > a$, centered at the origin, whereby

$$I_1 = \frac{1}{2} r_0 \int_{-\pi}^{\pi} [(t_{xx}u_x + t_{xy}u_y)_{r=r_0}\cos\theta + (t_{yy}u_y + t_{xy}u_x)_{r=r_0}\sin\theta] d\theta \tag{35}$$

The stress and displacement components obtained from Eqs. (5, 7, and 9) can be expanded as power series having the following forms:

$$t_{xx} = \frac{\tau}{2}\left\{2\sin 2\beta + \left(\frac{a}{r}\right)^2 [-2(\sin\theta\cos 3\theta + \sin 2\theta)\cos 2\beta + 2(\cos 2\theta - 2\sin\theta\sin 3\theta)\sin 2\beta] + f_x(\theta;\beta) O\left(\frac{a^4}{r^4}\right)\right\} \tag{36a}$$

$$t_{yy} = \frac{\tau}{2}\left\{-2\sin 2\beta + \left(\frac{a}{r}\right)^2 [2\sin\theta\cos 3\theta\cos 2\beta - (2\sin\theta\sin 3\theta + \cos 2\theta)\sin 2\beta] + f_y(\theta;\beta) O\left(\frac{a^4}{r^4}\right)\right\} \tag{36b}$$

$$t_{xy} = \frac{\tau}{2}\left\{2\cos 2\beta + \left(\frac{a}{r}\right)^2 [(\cos 2\theta - 2\sin\theta\sin 3\theta)\cos 2\beta - 2\sin\theta\cos 3\theta\sin 2\beta] + f_{xy}(\theta;\beta) O\left(\frac{a^4}{r^4}\right)\right\} \tag{36c}$$

and

$$u_x = \frac{\tau}{2\mu}\left\{r(\sin\theta\cos 2\beta + \cos\theta\sin 2\beta) + \frac{1}{4}\frac{a^2}{r}[\{(\kappa+1)\sin\theta + 2\sin\theta\cos 2\theta\}\cos 2\beta + \{(\kappa-1)\cos\theta - 2\sin\theta\sin 2\theta\}\sin 2\beta] + g_x(\theta;\beta) O\left(\frac{a^4}{r^3}\right)\right\} \tag{37a}$$

$$u_y = \frac{\tau}{2\mu}\left\{r(\cos\theta\cos 2\beta - \sin\theta\sin 2\beta) + \frac{1}{4}\frac{a^2}{r}[\{(\kappa-1)\cos\theta + 2\sin\theta\sin 2\theta\}\cos 2\beta - \{(\kappa+1)\sin\theta - 2\sin\theta\cos 2\theta\}\sin 2\beta] + g_y(\theta;\beta) O\left(\frac{a^4}{r^3}\right)\right\} \tag{37b}$$

where the functions $f_x, f_y, f_{xy}, g_x, g_y$ are sine and cosine functions in the variable angle θ and the parameter angle β. The condition for convergence of the preceding series requires $a/r < 1$. As $r \to \infty$, Eqs. (36) reduce to the boundary conditions (4); as $(2a) \to 0$, Eqs. (36) reduce to a uniform state of stress corresponding to the sheet without the crack. By means of Eqs. (36) and (37), integral I_1 assumes the form:

$$I_1 = \frac{\tau^2}{4\mu}\int_{-\pi}^{\pi}\left[r_0^2(\cos^2 2\beta + \sin^2 2\beta) + \frac{a^2}{4}F(\theta;\beta) + a^2\sum_{n=1}^{\infty}\left(\frac{a}{r_0}\right)^{2n} G_n(\theta;\beta)\right] d\theta \tag{38}$$

where

$$F(\theta;\beta) = (1 - 4\cos^2 2\beta)\sin^2 2\theta - (4\sin^2\theta\cos 4\beta + 2\cos^2\theta + 1)\cos 2\theta - 2(2\sin\theta\sin 3\theta + \sin 4\theta\sin 4\beta) + \kappa$$

and $G_n(\theta;\beta)$, $n = 1,2,\ldots$, represent functions composed of products and powers of sine and cosine functions. Integration of the first two terms of Eq. (38) leaves

$$I_1 = \frac{\tau^2\pi r_0^2}{2\mu} + \frac{\tau^2\pi a^2}{8\mu}(\kappa - 1) + \frac{\tau^2 a^2}{4\mu}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\left(\frac{a}{r_0}\right)^{2n} G_n(\theta;\beta) d\theta \tag{39}$$

As $I_2 = 0$, th strain energy stored in the infinite body is obtained entirely from I_1 in the limit $r_0 \to \infty$. The series in Eq. (39) may be shown to converge uniformly on the interval $[-\pi,\pi]$. For each n and all $\theta\epsilon[-\pi,\pi]$, the functions $G_n(\theta;\beta)$ are bound, while the sequence of positive constants $\{(a/r_0)^{2n}\}$ is a monotonically decreasing sequence that approaches zero as $n \to \infty$. Thus it is always possible to find the numbers M and N such that $|G_1(\theta;\beta) + \ldots + G_n(\theta;\beta)| < M$ for any $n > N$, implying uniform convergence by the Dirichlet test. It follows, therefore, that integration may be interchanged with summation, with the result vanishing in the limit, i.e.,

$$\lim_{r_0\to\infty}\left\{a^2\sum_{n=1}^{\infty}\left(\frac{a}{r_0}\right)^{2n}\int_{-\pi}^{\pi} G_n(\theta;\beta)d\theta\right\} = \lim_{r_0\to\infty}\left\{a^2\left[\left(\frac{a}{r_0}\right)^2 G_1^*(\beta) + \left(\frac{a}{r_0}\right)^4 G_2^*(\beta) + \ldots\right]\right\} = 0 \tag{40}$$

since each of the numbers $G_n^*(\beta)$ are bound quantities.

Of the remaining two contributions to I_1, the first is independent of the crack dimension and diverges in the limit, representing therefore the strain energy of the infinite sheet without the crack. The second is independent of the radial dimension r and is the only surviving term after passage to the limit that includes the crack dimension. It must therefore represent the change of elastic strain energy associated with the formation of a line crack having the dimension $2a$, independent of the orientation of the crack. In summary,

$$U = U_0 + \Delta U \tag{41}$$

where

$$U_0 \equiv \lim_{r_0\to\infty}(\tau^2\pi r_0^2/2\mu) \tag{42}$$

and

$$\Delta U \equiv \tau^2\pi a^2/8\mu(\kappa - 1) \tag{43}$$

The global rate of change of elastic strain energy with crack size, in this case the same for all crack orientations, is found by formal differentiation of Eq. (41)

$$\frac{\partial U}{\partial a} = \frac{\tau^2 \pi a}{4\mu} (\kappa - 1) \tag{44}$$

We note that the derivation leading to expressions (41-43) is direct and exact, employing only the stress and displacement fields uniquely required by the solution. Attention is called to this fact, perhaps superfluously, because of the difference appearing between Eq. (43) and the expression obtained quite circuitously and questionably by Sih and Liebowitz.[2,3] Specifically, the $(\kappa - 1)$ factor of Eq. (43) appears as $(\kappa + 1)$ in the Sih and Liebowitz formula. The method of analysis adopted by these authors appears to be questionable on the grounds that it employs "corrected" stress and displacement, rather than the stress and displacement uniquely specified by the exact solution in their calculation of the strain energy. The validity of Eq. (44) can be demonstrated by an alternative method of analysis, as shown in the Appendix.

VII. Critical Shear Stress

Griffith's crack instability criterion assumes that fracture of a body has occurred if the body can pass from the unbroken to the broken condition by a process in which the potential energy of the system continually decreases. The potential energy is defined to include a surface energy contribution, Γ, associated with crack surface formation. By means of Clapeyron's theorem, the potential energy reduces to the energy difference $(\Gamma - U)$. Γ is taken as a linear function of the crack surface area multiplied by an assumed constant surface energy density γ. Thus for unit thickness

$$\Gamma = 4\gamma a \tag{45}$$

The crack stability hypothesis requires as a necessary condition that at the point of incipient crack instability the quantity $(\Gamma - U)$ assume a stationary value, or

$$\frac{\partial}{\partial a}(\Gamma - U) = 0 \tag{46}$$

From Eqs. (41-43) and (45), condition (46) leads to the following expression for the critical value of the applied shear stress

$$\tau_{cr} = \sqrt{\frac{8E\gamma}{\pi a (1+\nu)(\kappa - 1)}} \tag{47}$$

which is independent of the angle of crack orientation.

Equation (47) coincides with the formula for pure shear loading obtained in an entirely different manner by Swedlow.[4] It differs from the formula of Sih and Liebowitz[2] in that the factor $(\kappa - 1)$ in the denominator of Eq. (47) appears as $(\kappa + 1)$ in their expression. For materials having aerospace applications, Poisson's ratio $\nu \approx 0.30$ and

$$\tau_{cr} = 0.87\sqrt{8E\gamma/\pi a} \quad \text{(plane stress)} \tag{48}$$

and

$$\tau_{cr} = 1.05\sqrt{8E\gamma/\pi a} \quad \text{(plane strain)} \tag{49}$$

The coefficient multiplying the square root factor in the corresponding Sih and Liebowitz formula are 0.50 for plane stress and 0.53 for plane strain, indicating differences by factors of 1.73 and 1.93, respectively, in the value of the critical applied shear stress.

Appendix

Consider the same bound region R shown by Fig. 11, but now having a crack size $2(a+\Delta a)$, where the increment Δa may be arbitrarily small. Let t'_{jk}, e'_{jk}, and u'_k, $j,k=1,2$ (where $t'_{11} = t'_{xx}$, $t'_{12} = t'_{xy}$, $u'_1 = u'_x$,..., etc.) designate the stress, strain, and displacement components for the body with the extended crack. The applied tractions and the displacements along the remote boundaries remain as before, since crack size variation will not alter boundary stress and displacement for an infinite body (as may be verified by Eqs. (36) and (37) with the variable r taken infinitely large). For a body having finite dimensions, this will, in general, not be true. The solution for the body with the extended crack is therefore identical in all respects with the solution corresponding to the original crack, with the exception that the crack size parameter $2a$ is replaced by $2(a+\Delta a)$. The elastic strain energy per unit thickness over the bound region R for the body with the extended crack is:

$$U(R, a+\Delta a) = U'(R) = \frac{1}{2}\iint_R t'_{jk} e'_{jk}\,dA = \frac{1}{2}\iint_R \left[t'_{xx}\frac{\partial u'_x}{\partial x} + t'_{yy}\frac{\partial u'_y}{\partial y} + t'_{xy}\left(\frac{\partial u'_x}{\partial y} + \frac{\partial u'_y}{\partial x}\right)\right] dx dy \tag{A1}$$

where the usual summation convection for index notation is implied. The difference in strain energy over the region R with the crack having dimensions $2(a+\Delta a)$ and $2a$, respectively, is, therefore,

$$U'(R) - U(R) = \frac{1}{2}\iint_R [t'_{jk}e'_{jk} - t_{jk}e_{jk}]\,dA \tag{A2}$$

By means of the identity (cf. Ref. 8 for a proof)

$$t'_{jk}e'_{jk} - t_{jk}e_{jk} = (t'_{jk} + t_{jk})(e'_{jk} - e_{jk}) = (t'_{jk} - t_{jk})(e'_{jk} + e_{jk}) \tag{A3}$$

the strain energy difference, Eq. (A2), may be expressed alternatively as

$$U'(R) - U(R) = \frac{1}{2}\iint_R (t'_{jk} + t_{jk})(e'_{jk} - e_{jk})\,dA = \frac{1}{2}\iint_R (t'_{jk} - t_{jk})(e'_{jk} + e_{jk})\,dA \tag{A4}$$

Transforming the first double integral to a line integral once again by means of Green's theorem (for the multiply connected domain)

$$\begin{aligned} U'(R) - U(R) &= [U'(R) - U(R)]_{C_1} + [U'(R) - U(R)]_{C_2} \\ &= \frac{1}{2}\oint_{C_1} \{[(t'_{xx} + t_{xx})(u_x - u_x) + (t'_{xy} + t_{xy})(u'_y - u_y)]dy \\ &\quad - [(t'_{yy} + t_{yy})(u'_y - u_y) + (t'_{xy} + t_{xy})(u'_x - u_x)]dx\} \\ &\quad + \frac{1}{2}\oint_{C_2} \{[(t'_{xx} + t_{xx})(u'_x - u_x) + (t'_{xy} + t_{xy})(u'_y - u_y)]dy \\ &\quad - [(t'_{yy} + t_{yy})(u'_y - u_y) + (t'_{xy} + t_{xy})(u'_x - u_x)]dx\} \end{aligned} \tag{A5}$$

The boundary C_2 may again be taken to coincide with the borders of the crack, but now having the length $2(a+\Delta a)$. Designating the upper crack surface as positive and the lower as negative, a little reflection will show that by virtue of symmetry considerations and the boundary conditions Eq.

(6), the second line integral of Eq. (A5) reduces to

$$[U'(R)-U(R)]_{C_2}=-4\int_a^{a+\Delta a}\frac{1}{2}\,(t_{yy}^{(+)}u_y'^{(+)}+t_{xy}^{(+)}u_x'^{(+)})\,dx \tag{A6}$$

where it is recalled that $t_{yy}^{(+)}$ and $t_{xy}^{(+)}$ refer to the body with the original crack and $u_y'^{(+)}$ and $u_x'^{(+)}$ refer to the body with the extended crack. Equation (A6) represents the equivalent of the work performed along the upper and lower faces of the crack in extending both crack ends by the increments Δa. A circle of arbitrarily large radius $r_0>a$, centered at the origin, is again most convenient for the choice of the outer boundary contour C_1. With these particular choices, the strain energy difference (A5) assumes the form

$$\begin{aligned}U'(R)-U(R)&=[U'(R)-U(R)]_{C_1}+[U'(R)-U(R)]_{C_2}\\&=\frac{1}{2}\int_{-\pi}^{\pi}\{[(t_{xx}'+t_{xx})(u_x'-u_x)\\&+(t_{xy}'+t_{xy})(u_y'-u_y)]_{r=r_0}\cos\theta+[(t_{yy}'+t_{yy})(u_y'-u_y)\\&+(t_{xy}'+t_{xy})(u_x'-u_x)]_{r=r_0}\sin\theta\}r_0\,d\theta\\&-4\int_a^{a+\Delta a}\frac{1}{2}\,(t_{yy}^{(+)}u_y'^{(+)}+t_{xy}^{(+)}u_x'^{(+)})\,dx\end{aligned} \tag{A7}$$

From Eqs. (36) it may be seen that when terms of higher order in the arbitrarily small increment of crack extension Δa are ignored, then along the circular boundary C_1

$$(t_{yy}'+t_{yy})=-\tau\Big\{2\sin2\beta+\frac{a^2+a\cdot\Delta a}{r_0^2}[2\sin\theta\cos3\theta\cos2\beta+(\cos2\theta-2\sin\theta)\sin2\beta]+f_y(\theta;\beta)O\Big(\frac{a^4}{r_0^4}\Big)\Big\} \tag{A8a}$$

$$(t_{xx}'+t_{xx})=\tau\Big\{2\sin2\beta+\frac{a^2+a\cdot\Delta a}{r_0^2}[2(\sin\theta\cos3\theta-\sin2\theta)\cos2\beta-(2\sin\theta\sin3\theta+\cos2\theta)\sin2\beta]+f_x(\theta;\beta)O\Big(\frac{a^4}{r_0^4}\Big)\Big\} \tag{A8b}$$

$$(t_{xy}'+t_{xy})=\tau\Big\{2\cos2\beta+\frac{a^2+\Delta a}{r_0^2}[(\cos2\theta+2\sin\theta\sin3\theta)\cos2\beta+2\sin\theta\cos3\theta\sin2\beta]+f_{xy}(\theta;\beta)O\Big(\frac{a^4}{r_0^4}\Big)\Big\} \tag{A8c}$$

while from Eqs. (37)

$$(u_x'-u_x)=\frac{\tau}{4\mu}\Big\{\frac{a\cdot\Delta a}{r_0}[\langle(\kappa+1)\sin\theta-2\sin\theta\cos2\theta\rangle\cos2\beta+\langle(\kappa-1)\cos\theta+2\sin\theta\sin2\theta\rangle\sin2\beta]+g_x(\theta;\beta)O\Big(\frac{a^3}{r_0^3}\Big)\Big\} \tag{A9a}$$

$$(u_y'-u_y)=\frac{\tau}{4\mu}\Big\{\frac{a\cdot\Delta a}{r_0}[\langle(\kappa-1)\cos\theta-2\sin\theta\sin2\theta\rangle\cos2\beta-\langle(\kappa+1)\sin\theta+2\sin\theta\cos2\theta\rangle\sin2\beta]+g_y(\theta;\beta)O\Big(\frac{a^3}{r_0^3}\Big)\Big\} \tag{A9b}$$

Substitution of these expressions into the first integral of Eq. (A7) yields

$$\begin{aligned}[U'(R)-U(R)]_{C_1}&=\frac{\tau^2}{4\mu}\int_{-\pi}^{\pi}\{a\cdot\Delta a[\kappa+\sin^2\theta-\cos^2\theta\\&+(\sin^22\beta-\cos^22\beta)(2\cos2\theta\sin^2\theta+\sin^22\theta)+\sin4\beta\\&\times(2\sin2\theta\sin^2\theta-½\sin4\theta)]+a\cdot\Delta a\sum_{n=1}^{\infty}\Big(\frac{a}{r_0}\Big)^{2n}G_n(\theta;\beta)\}d\theta\end{aligned} \tag{A10}$$

The functions $G_n(\theta;\beta)$ are again sums and products of sine and cosine functions so that, as before, the series may be shown to converge uniformly over $[-\pi,\pi]$. Thus,

$$[U'(R)-U(R)]_{C_1}=\frac{\tau^2\pi a}{4\mu}2\kappa\cdot\Delta a+\frac{\tau^2 a}{4\mu}\cdot\Delta a\sum_{n=1}^{\infty}\Big(\frac{a}{r_0}\Big)^{2n}\int_{-\pi}^{\pi}G_n(\theta;\beta)\,d\theta \tag{A11}$$

where it is clear that after term-by-term integration, the resulting sum will vanish in the limit as $r_0\to\infty$. Since Δa is arbitrarily small, evaluation of the second integral of Eq. (A7) can be simplified by shifting the origin from the crack center to the crack tip and using the crack-tip expressions, Eqs. (16-20), for the stress and displacement. From Eqs. (16) we have

$$[t_{yy}^{(+)}]_{\substack{\theta=0\\r=x}}=-\frac{\tau\sqrt{a}}{\sqrt{2x}}\sin2\beta,\quad [t[_{xy}^{(+)}]_{\substack{\theta=0\\r=x}}=\frac{\tau\sqrt{a}}{\sqrt{2x}}\cos2\beta \tag{A12}$$

for the unextended crack, while from Eq. (20),

$$[u_y'^{(+)}]_{\substack{\theta=\pi\\r=(\Delta a-x)}}=-\frac{\tau\sqrt{\pi(a+\Delta a)}\sin2\beta(\kappa+1)}{2\mu}\sqrt{\frac{\Delta a-x}{2\pi}}+\frac{\tau}{4\mu}(\kappa+1)(x+a)\cos2\beta \tag{A13a}$$

$$[u_x'^{(+)}]_{\substack{\theta=\pi\\r=(\Delta a-x)}}=\frac{\tau\sqrt{\pi(a+\Delta a)}\cos2\beta(\kappa+1)}{2\mu}\sqrt{\frac{\Delta a-x}{2\pi}}+\frac{\tau}{4\mu}(\kappa+1)(x+a)\sin2\beta \tag{A13}$$

for the extended crack. The second integral of Eq. (A7) follows as

$$[U'(R)-U(R)]_{C_2}=-\frac{\tau^2a(\kappa+1)}{2\mu}\int_0^{\Delta a}\sqrt{\frac{\Delta a-x}{x}}\,dx=-\frac{\tau^2\pi a(\kappa+1)}{4\mu}\cdot\Delta a \tag{A14}$$

By means of Eqs. (A11) and (A14), the elastic strain energy rate for the infinite body is obtained in the limits $r_0\to\infty$ and $\Delta a\to0$; that is,

$$\frac{\partial U}{\partial a}=\lim_{\substack{r_0\to\infty\\\Delta a\to0}}\Big\{\Big[\frac{U'(R)-U(R)}{\Delta a}\Big]_{C_1}+\Big[\frac{U'(R)-U(R)}{\Delta a}\Big]_{C_2}\Big\}=\frac{\tau^2\pi a}{4\mu}(2\kappa)-\frac{\tau^2\pi a}{4\mu}(\kappa+1)=\frac{\tau^2\pi a}{4\mu}(\kappa-1) \tag{A15}$$

which reproduces Eq. (44).

This particular way of determining the rate of change of the elastic strain energy with crack size has the advantage of showing how it can be decomposed into two contributions. The first corresponds to the rate of mechanical work of the applied boundary tractions. The second corresponds to the mechanical work performed by the stresses in the process of crack extension at both crack ends, and is usually referred to in the literature as the "strain energy release rate" with the designated symbol G. This "strain energy release rate" is often confused with the total elastic strain energy rate of the body, thereby either ignoring the first contribution or implying that it is not significant, contrary to what is shown by Eq. (A15). In fact, examination of either Eqs. (A4) or (A5) plainly shows that the "strain energy release rate" G can represent the total elastic strain energy rate, only if a boundary constraint is arbitrarily imposed on the crack extension process, e.g., a "fixed-grips" or a "dead-load" boundary condition. The importance of recognizing this distinction particularly manifests itself in calculating the fracture stress, which requires use of the total elastic strain energy rate, i.e., Eqs. (44) or (A15).

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